A Relation between θ_{13} and the Leptonic Dirac CP Phase in $SO(10)$ Lopsided Models

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It is shown that in $SO(10)$ models where the large solar and atmospheric neutrino angles come from the charged-lepton mass matrix being "lopsided", there is a characteristic relation between the 13 mixing angle of the neutrinos and the size of the Dirac CP- violating phase in the lepton sector. This is illustrated in a recently proposed realistic and predictive model.

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The symmmetries of the Standard Model do not constrain the masses and mixings of the quarks and leptons, which are therefore free parameters of that model. The best hope for obtaining predictions (and testably precise postdictions) of these quantities seems to lie with the more powerful symmetries of grand unified theories. The greatest degree of predictivity comes from the unification group $SO(10)$, since it relates the mass matrices of the up-type quarks, down-type quarks, and charged leptons, and the Dirac mass matrix of the neutrinos. (These four mass matrices will be denoted here by M_U, M_D, M_L, and M_N, respectively.) An obstacle to making predictions, however, is the fact that in the usual "type I see-saw" set-up [1] the mass matrix of the observed light neutrinos M_ν depends on the Majorana mass matrix of the right-handed neutrinos M_R through the well-known see-saw formula $M_\nu = -M_N M_R^{-1} M_N^T$. And because M_R tends in most models to be only loosely related by SO(10) symmetry, if at all, to the other mass matrices, our experimental knowledge of the properties of the quarks and charged leptons gives no information about M_R. Since M_R is unconstrained and is a symmetric three-by-three complex matrix, it introduces many free parameters into the calculation of the light neutrino masses and mixing angles.

In this letter, we discuss a simple class of $SO(10)$ models, well-motivated on other grounds, in which the number of free parameters coming from M_R is much reduced and where it is therefore possible to get definite predictions for the neutrino mixing matrix U_{MNS} [2], including a testable relation between two Standard Model quantities that have not yet been measured, namely θ_{13} and δ_{lep}. (We denote by δ_{lep} the "Dirac" CP-violating phase of the lepton sector. For a review of leptonic CP violation, see [3].)

The neutrino mixing matrix is given by the standard formula $U_{MNS} = U_L U_\nu^\dagger$, where U_L is the unitary transformation of the left-handed charged leptons required to diagonalize the charged-lepton mass matrix M_L, and U_ν is the unitary matrix that diagonalizes the mass matrix of the observed light neutrinos M_ν. Typically, the matrix U_L is highly constrained or even known in $SO(10)$ models, because the unified symmetry relates M_L to the quark mass matrices; but in most models U_ν is poorly constrained or unknown, because of its dependence on M_R. What is different about the models we are discussing in this paper is that the neutrino Dirac mass matrix M_N is assumed to have negligibly small elements in its first row and column (in the "original basis" defined by the flavor symmetries of the model). This obviously implies that $M_\nu = -M_N M_R^{-1} M_N^T$ also has negligibly small first row and column, which means that U_ν is in effect a $U(2)$ rather than a $U(3)$ rotation. As such, it contains only one real rotation angle and three complex phases, of which two phases do not contribute to low-energy physics. In other words, in the kind of model we shall discuss, only two parameters that depend on M_R actually come into the computation of the leptonic mixing matrix U_{MNS}, namely one angle and one phase that we shall call θ_ν and ϕ_ν.

The crucial assumption that the first row and column of M_N are very small is motivated by two observations. First, in $SO(10)$ models there tends to be a close relationship between M_N and the mass matrix of the up-type quarks M_U. Second, in many models M_U has very small elements in its first row and column to account for the extreme smallness of its smallest eigenvalue compared to its largest: $m_u/m_t \sim 10^{-5}$, which is much less than the corresponding ratios for M_D and M_L: $m_d/m_b \sim 10^{-3}$ and $m_e/m_\tau \sim 0.3 \times 10^{-3}$.

If, as we are assuming, U_ν is to a good approximation a $U(2)$ rotation of the second and third families, then the large solar neutrino angle, which involves the first family, must come from the U_L rather than from U_ν. That is, the solar neutrino angle must come from the diagonalization of M_L rather than M_ν. But this means that in the original basis M_L has large off-diagonal elements, which is the distinguishing feature of so-called "lopsided models" [4, 5, 6, 7]. In particular, one is naturally led to models of the "doubly lopsided" form [4, 8, 9, 10, 11].

The basic idea of "lopsided models" is that large neutrino mixing angles are caused by large asymmetrical off-diagonal elements in M_L. All lopsided models explain the large atmospheric angle by the 23 element of M_L being large, i.e. as large as the 33 element. In doubly lopsided models the 13 element of M_L is also assumed to be large to explain the large solar angle. If these large elements arise in an $SU(5)$-

invariant way (i.e. from effective operators of the form $C_1\mathbf{10}_1\overline{\mathbf{5}}_3\langle\overline{\mathbf{5}}_H\rangle + C_2\mathbf{10}_2\overline{\mathbf{5}}_3\langle\overline{\mathbf{5}}_H\rangle + C_3\mathbf{10}_3\overline{\mathbf{5}}_3\langle\overline{\mathbf{5}}_H\rangle$), then the matrices M_L and M_D have the form

$$M_L = \begin{pmatrix} - & - & C_1 \\ - & - & C_2 \\ - & - & C_3 \end{pmatrix} v_d, \quad M_D = \begin{pmatrix} - & - & - \\ - & - & - \\ C_1 & C_2 & C_3 \end{pmatrix} v_d \tag{1}$$

where the dashes indicate elements much smaller than the C_i. (The convention we use is that the left-handed fermions multiply the mass matrix from the left, and the right-handed fermions multiply it from the right.) The forms in Eq. (1) reflect the well-known fact that $SU(5)$ relates M_L to the transpose of M_D. The reason for this left-right transposition is that the $\mathbf{10}$'s of $SU(5)$ contain the left-handed down-type quarks and right-handed charged leptons, while the $\overline{\mathbf{5}}$'s contain the right-handed down-type quarks and left-handed charged leptons. That is why the large lopsided mass-matrix elements $C_{1,2}$ produce large mixing of the *left*-handed leptons but of the *right*-handed quarks, which accounts for the fact that the MNS angles are big and the CKM angles are small.

The large elements of M_L can be diagonalized by two successive rotations of the left-handed charged leptons:

$$\begin{pmatrix} - & - & C_1 \\ - & - & C_2 \\ - & - & C_3 \end{pmatrix} \overset{U_{12}(\theta_s)}{\longrightarrow} \begin{pmatrix} - & - & 0 \\ - & - & \sqrt{|C_1|^2+|C_2|^2} \\ - & - & C_3 \end{pmatrix} \overset{U_{23}(\theta_a)}{\longrightarrow} \begin{pmatrix} - & - & 0 \\ - & - & 0 \\ - & - & C \end{pmatrix} \tag{2}$$

where $C \equiv \sqrt{|C_1|^2+|C_2|^2+|C_3|^2}$, $\tan\theta_s = C_1/C_2$ and $\tan\theta_a = \sqrt{C_1^2+C_2^2}/C_3$. Another rotation of the left-handed charged leptons (call it $U'_{12}(\eta)$) is required to eliminate the small 12 element that remains after the first two rotations. (The small elements that remain *below* the main diagonal are eliminated by rotations of the *right-handed* leptons.) Thus U_L has the form $U_L = U'_{12}(\eta)U_{23}(\theta_a)U_{12}(\theta_s)$.

The magnitude of the third rotation angle, η, depends on the relative magnitudes of the small elements of M_L that are denoted by dashes in Eq. (1). If, as in the models we shall be considering, the 32 element of M_L is much larger than the 22 and 12 elements, then the angle η is small, and U_L has the approximate form

$$\begin{aligned} U_L &\cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_a & s_a \\ 0 & -s_a & c_a \end{pmatrix} \begin{pmatrix} c_s & s_s & 0 \\ -s_s & c_s & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_s & s_s & 0 \\ -c_a s_s & c_a c_s & s_a \\ s_a s_s & -s_a c_s & c_a \end{pmatrix} \end{aligned} \tag{3}$$

where $s_a \equiv \sin\theta_a$, $c_a \equiv \cos\theta_a$, $s_s \equiv \sin\theta_s$, and $c_s \equiv \cos\theta_s$. If the angles in U_ν are small (as they will be in the models we are considering, because of the hierarchical nature of M_ν), then $U_{MNS} = U_L U_\nu^\dagger$ will be approximately given by Eq. (3). One sees from this that the doubly lopsided structure accounts in a simple and natural way for the "bi-large" form of U_{MNS}, i.e. the form in which the solar and atmospheric angles are large, but the 13 mixing, U_{e3} is small. Note, however, that the angles θ_a and θ_s in Eq. (3) are not exactly equal to the atmospheric and solar neutrino angles, which we will denote by θ_{atm} and θ_{sol}, since the latter get small contributions from η and from the angles in U_ν.

In the models we are discussing, U_L, can be determined from the known quark masses, charged lepton masses and the CKM angles. That means that U_{MNS} depends only on the two unknown parameters θ_ν and ϕ_ν coming from U_ν. Since the solar neutrino angle θ_{sol} tends to be quite insensitive to these parameters, as will be seen, one has three observable quantities in U_{MNS}, (θ_{atm}, θ_{13}, and δ_{lep}) being calculable in terms of just two free parameters, thus yielding one prediction, which can be expressed as a relation between θ_{13} and δ_{lep}.

To see what kind of relation one expects, let us neglect η and approximate U_L by the simple form in Eq. (3). Then

$$U_{MNS} \cong \begin{pmatrix} c_s & s_s & 0 \\ -c_a s_s & c_a c_s & s_a \\ s_a s_s & -s_a c_s & c_a \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\nu & s_\nu e^{i\phi_\nu} \\ 0 & -s_\nu e^{-i\phi_\nu} & c_\nu \end{pmatrix}. \tag{4}$$

Here, $s_\nu \equiv \sin\theta_\nu$ and $c_\nu \equiv \cos\theta_\nu$. Multiplying this out, one obtains (for small θ_ν):

$$\begin{aligned} \sin\theta_{sol} &\cong c_\nu s_s \cong \sin\theta_s, \\ \sin\theta_{atm} &\cong |c_\nu s_a + s_\nu c_a c_s e^{i\phi_\nu}| \\ &\cong \sin\theta_a + \cos\theta_a \cos\theta_{sol} \sin\theta_\nu \cos\phi_\nu, \\ \sin\theta_{13} &\cong s_s s_\nu \cong \sin\theta_{sol}\sin\theta_\nu, \\ \delta_{lep} &\cong \phi_\nu. \end{aligned} \tag{5}$$

The last of these equations results from the fact that the 13 element of U_{MNS} has a phase ϕ_ν, whereas the phases of the other elements appear in terms that are subleading in the small quantity $\sin\theta_\nu$. We may rewrite the second equation in Eq. (5) as

$$\sin\theta_\nu \cong \frac{\Delta}{\cos\phi_\nu}, \quad \Delta \equiv \frac{\sin\theta_{atm} - \sin\theta_a}{\cos\theta_a \cos\theta_{sol}}. \tag{6}$$

Therefore

$$\sin\theta_{13} \cong \frac{\sin\theta_{sol}\Delta}{\cos\delta_{lep}}. \tag{7}$$

In realistic doubly lopsided models based on $SO(10)$ [5, 11], it is typically found by fitting the quark masses and mixing angles that $\theta_a \sim \pi/3$, so that $\Delta \sim 0.25$. A graph of Eq. (7) using this value is shown in Fig. 1.

We now illustrate these ideas in a particular model that is both predictive and realistic model [9, 11]. It is a non-supersymmetric $SO(10)$ grand unified model, in which

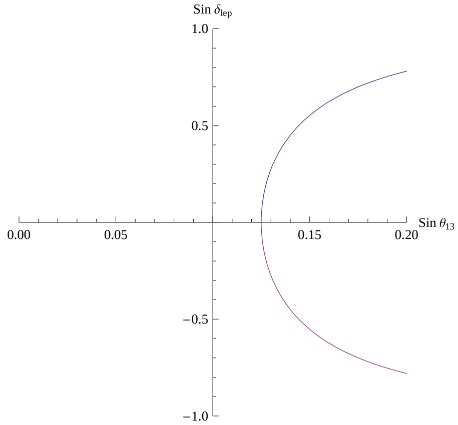

FIG. 1: The relation between $\sin\delta_{lep}$ and $\sin\theta_{13}$ given in Eq. 7 for $\Delta = 0.25$.

the mass hierarchy among the families arises radiatively; i.e. the masses of the second and third families arise at tree level and the masses of the first family from loop diagrams. The details of this model are set forth in other papers [9, 11]; here, we merely summarize. The mass matrices in this model have the approximate form (we use slightly different notation than in [11]):

$$\frac{M_U}{v_u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{\epsilon}{3} \\ 0 & -\frac{\epsilon}{3} & 1 \end{pmatrix}, \quad \frac{M_D}{v_d} = \begin{pmatrix} 0 & 0 & \delta \\ 0 & \delta_H & \frac{\epsilon}{3}+\delta' \\ fC_1 & fC_2 - \frac{\epsilon}{3} & 1 \end{pmatrix}$$

$$\frac{M_N}{v_u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \quad \frac{M_L}{v_d} = \begin{pmatrix} 0 & 0 & C_1 \\ 0 & f_H\delta_H & C_2 - \epsilon \\ 3\delta & \epsilon + 3\delta' & 1 \end{pmatrix} \tag{8}$$

where $\delta' \equiv (C_2/C_1)\delta$. The parameters denoted by ϵ, δ, and δ_H are small, so that M_L and M_D have the forms given in Eq. (1) (with v_d and v_u scaled to make $C_3 = 1$), and M_U and M_N indeed have approximately vanishing first row and column.

The elements in these matrices denoted by 1, ϵ, and C_i, $i = 1, 2$ arise at tree-level from three effective operators: $O_1 = \mathbf{16}_3\mathbf{16}_3\mathbf{10}_H$, $O_2 = \mathbf{16}_2\mathbf{16}_3\mathbf{10}_H\mathbf{45}_H/M_2$, and $O_3 = c_i\mathbf{16}_i\mathbf{16}_3\mathbf{16}_{iH}\mathbf{16}'_H/M_3$ (i=1,2), respectively. Some of the structure of these tree-level elements is easily understood group-theoretically. The vacuum expectation value (VEV) of the adjoint Higgs field $\langle\mathbf{45}_H\rangle$ in O_2 is proportional to the $SO(10)$ generator $B-L$ and gives the factor of $-\frac{1}{3}$ in the ϵ terms of the quark matrices relative to the lepton matrices. That factor is responsible for the well-known Georgi-Jarlskog relation of quark to lepton masses [12]. In O_3, the fact that the Higgs fields are in spinors ($\mathbf{16}$) of $SO(10)$, which contain $\overline{\mathbf{5}}$ but not $\mathbf{5}$ of $SU(5)$, explains why this operator contributes the elements C_i only to M_D and M_L, and not to M_U or M_N.

The elements denoted by δ, δ', and δ_H arise from one-loop diagrams. f and f_H are factors reflecting the breaking of $SO(10)$ (or more precisely, of $SU(4)_c$). The parameters v_d and v_u set the overall scales of the mass matrices of the $I_3 = -\frac{1}{2}$ and $I_3 = +\frac{1}{2}$ fermions, and have a small ratio ($v_d/v_u \simeq 0.9\times10^{-2}$) that is responsible for the small ratio of m_b to m_t. (v_d and v_u come respectively from the VEVs of the $SU(5)$ $\overline{\mathbf{5}}$ and $\mathbf{5}$ in the $SO(10)$ $\mathbf{10}$ of Higgs fields.) Aside from this ratio of VEVs, all dimensionless parameters of the model that come into the quark and lepton mass ratios and mixing angles are of order 1 if they arise at tree-level, and of order $1/16\pi^2$ if they arise at one-loop level: a fit of the data [11] gives $\epsilon \cong 0.189$, $C_1 \cong 1.03$, $C_2 \cong -1.51$, $f \cong 0.566$, $f_H \cong 0.208$, $\delta \cong 2.29(16\pi^2)^{-1}$, and $\delta_H \cong 2.66(16\pi^2)^{-1}$. Some of these parameters are complex. There are four physical phases, but of these only two have a significant effect on the fit, and these also are of order 1: $\arg(\epsilon) \cong 1.52$ rad and $\arg(\delta_H) \cong 0.514$ rad.

In spite of the fact that the dimensionless parameters of the model have "natural" values, there is a good fit with 11 parameters to 14 measured quantities that span a very wide range: namely the quark masses, charged lepton masses, CKM parameters, and the solar and atmospheric angles. Of course, from the four Dirac mass matrices in Eq. (8), it is not the angles in U_{MNS} that are predicted, but the angles in U_L. From the fit to the data performed in [11], the best fit value of $(U_L)_{23}$ comes out to be 0.891, whereas the experimental central value of $(U_{MNS})_{23} \equiv \sin\theta_{atm}$ is about 0.71.

Since the matrices M_U and M_N in Eq. (8) have vanishing first row and column, the mass of the up quark is zero at this level. The up quark mass can be fit by a 11 element of M_U/v_u that is of order 10^{-5}. That is too small to be a one-loop effect, but it is the right magnitude to be a two-loop or three-loop effect. One expects from $SO(10)$ symmetry that in M_N/v_u there would also be a 11 element of order 10^{-5}. That should have negligible effect on the predictions for the neutrino mixing parameters that will be presented below.

To obtain predictions for the angles θ_{13} and δ_{lep}, we fix the parameters appearing in Eq. (8) to the values that give the best χ^2 fit to the following set of measured parameters: quark masses, CKM angles, CKM phase, charged lepton masses, and solar neutrino angle. This numerical fit was done in [11], and the details can be found in that paper. We then scan over the possible values of θ_ν and ϕ_ν for those that give a particular value of the atmospheric neutrino angle θ_{atm} and plot the resulting points in the θ_{13}-δ_{lep} plane. This is shown in Fig. 2, where the best-fit points coalesce to form the dark curves in the center of the shaded bands. These curves, as expected, are similar to the one shown in Fig. 1. Each shaded band represents a different value of $\sin^2\theta_{atm}$. The central curve in each band corresponds to the parameters that give the best χ^2 for the set of measured parameters, which is 4.5. The shaded region corresponds to fits for which $\chi^2 \leq 6.5$.

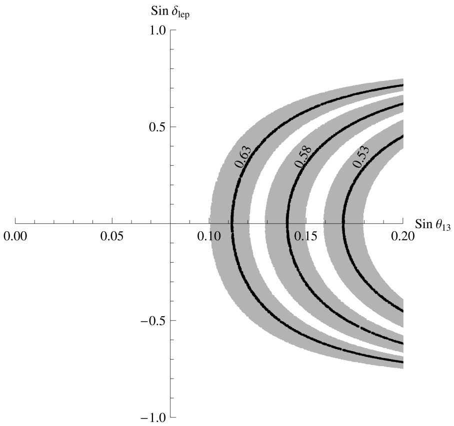

FIG. 2: The actual relation between $\sin\delta_{lep}$ and $\sin\theta_{13}$ for various values of $\sin^2\theta_{atm}$ (the numbers in the shaded band) in a realistic model

The major source of the uncertainty shown by the shaded bands is the phase of the parameter f_H in Eq. (8). This phase has no direct effect on the quark masses and mixing angles and a relatively weak effect on most of the leptonic quantities due to the fact that f_H is such a small parameter. (Of course, $\arg f_H$ does have an *indirect* effect on the quark masses and mixing angles, since the effects on the very precisely known charged lepton masses from varying $\arg f_H$ have to be compensated in the χ^2 fit by changes in the other parameters.) Since $\arg f_H$ can vary considerably without harming the χ^2 fit to the other quantities, it has a significant effect on θ_{13} and δ_{lep}.

In conclusion, doubly lopsided models based on $SO(10)$ in which the Dirac neutrino mass matrix has very small first row and column can give interesting and testable predictions for the two as-yet-unknown parameters of the Standard Model, θ_{13} and δ_{lep}. In particular, there is a fairly precise relation between these two quantities, such that the smaller θ_{13} is the smaller is δ_{lep}, with a lower limit on θ_{13}, as Figs. 1 and 2 show. These features have been illustrated in a particular realistic model, which is non-supersymmetric and has a radiative fermion mass hierarchy. However, qualitatively similar predictions should also be obtainable from doubly lopsided $SO(10)$ models that are supersymmetric and that have tree-level hierarchies. The predicted relation between θ_{13} and δ_{lep} will become more precise as the quark masses, CKM angles, CKM phase, and solar and atmospheric neutrino angles are determined with more precision. If they were known perfectly, the predicted relation would be a single sharp curve like the one shown in Fig. 1. One sees, then, that the rigorous testing of models of quark and lepton masses will require progress along a broad front.

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